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HOMOGENOUS SECOND ORDER ODE

Summary

- Homogenous Second order ODE with constant coefficient is solved by auxiliary equation (characteristic equation).
- A second solution (y_2) of Homogenous Second order DE with non-constant coefficient can be solved by reduction of order; provided that a solution (y_1) is firstly known.

Homogenous 2nd Order DE with constant coefficient

The general form of the second order differential equation with constant coefficients is

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = Q(x) \qquad \equiv \qquad ay'' + by' + cy = Q(t)$$

For the homogenous second order differential equation is

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0 \qquad \equiv \qquad ay'' + by' + cy = 0$$

With a, *b* and *c* are constants.

Homogenous 2nd Order DE with constant coefficient

The equation:

$$am^2 + bm + c = 0$$

is called the **Auxiliary Equation** (A.E.) or **Characteristic Equation**.

The **general solution** of the differential equation depends on the solution of the A.E. To find the general solution, we must determine the roots of the A.E. The roots of the A.E. are given by the well-known quadratic formula:

ABC Formula →

$$m_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Summary of general solution for homogenous 2nd order DE with constant coefficient:

Case	Roots of A.E.	Condition	General solution
1	Real distinct roots: m_1 and m_2	$b^2 - 4ac > 0$	$y = Ae^{m_1 t} + Be^{m_2 t}$
2	Real repeated roots: m	$b^2 - 4ac = 0$	$y = e^{mt}(A + Bt)$
3	Complex conjugate roots: $m_1 = \alpha + j\omega$ $m_2 = \alpha - j\omega$	$b^2 - 4ac < 0$	$y = e^{\alpha t}(A\cos\omega t + B\sin\omega t)$

Example

Solve the following differential equations

1. $$\frac{d^2y}{dx^2}-5\frac{dy}{dx}+4y=0$$

2. $$\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=0$$

3. $$y''+4y=0$$

Solution

1. $$\frac{d^2y}{dx^2}-5\frac{dy}{dx}+4y=0$$

Writing this equation in the form of characteristic equation, we know that,

$$m^2-5m+4=0$$

We get → $b^2-4ac>0$, so the roots are real and distinct: $(m-4)(m-1)=0$

The roots are $m_1=1, m_2=4$

The general solution is: $y=Ae^{x}+Be^{4x}$

Example:

Solve the following initial value problem

$$y'' + 11y' + 24y = 0 \qquad y(0) = 0 \qquad y'(0) = -7$$

Solution

The characteristic equation is

$$r^2 + 11r + 24 = 0$$

$$(r+8)(r+3) = 0$$

Its roots are $r_1 = -8$ and $r_2 = -3$ and so the general solution and its derivative is.

$$y(t) = c_1\mathbf{e}^{-8t} + c_2\mathbf{e}^{-3t}$$

$$y'(t) = -8c_1\mathbf{e}^{-8t} - 3c_2\mathbf{e}^{-3t}$$

Now, plug in the initial conditions to get the following system of equations.

$$0 = y(0) = c_1 + c_2$$

$$-7 = y'(0) = -8c_1 - 3c_2$$

Solving this system gives $c_1 = \frac{7}{5}$ and $c_2 = -\frac{7}{5}$. The actual solution to the differential equation is then

$$y(t) = \frac{7}{5}\mathbf{e}^{-8t} - \frac{7}{5}\mathbf{e}^{-3t}$$

Example 2:

$$y'' - 8y' + 17y = 0 \qquad y(0) = -4 \quad y'(0) = -1$$

Example 3:

$$y'' - 4y' + 4y = 0 \qquad y(0) = 12 \qquad y'(0) = -3$$

$$y'' - 8y' + 17y = 0 \qquad y(0) = -4 \quad y'(0) = -1$$

Solution

The characteristic equation this time is.

$$r^2 - 8r + 17 = 0$$

The roots of this are $r_{1,2} = 4 \pm i$. The general solution as well as its derivative is

$$y(t) = c_1 \mathbf{e}^{4t} \cos(t) + c_2 \mathbf{e}^{4t} \sin(t)$$

$$y'(t) = 4c_1 \mathbf{e}^{4t} \cos(t) - c_1 \mathbf{e}^{4t} \sin(t) + 4c_2 \mathbf{e}^{4t} \sin(t) + c_2 \mathbf{e}^{4t} \cos(t)$$

Notice that this time we will need the derivative from the start as we won't be having one of the terms drop out. Applying the initial conditions gives the following system.

$$-4 = y(0) = c_1$$

$$-1 = y'(0) = 4c_1 + c_2$$

Solving this system gives $c_1 = -4$ and $c_2 = 15$. The actual solution to the IVP is then.

$$y(t) = -4\mathbf{e}^{4t} \cos(t) + 15\mathbf{e}^{4t} \sin(t)$$

$$y'' - 4y' + 4y = 0 \qquad y(0) = 12 \qquad y'(0) = -3$$

Solution

The characteristic equation and its roots are.

$$r^2 - 4r + 4 = (r-2)^2 = 0 \qquad r_{1,2} = 2$$

The general solution and its derivative are

$$y(t) = c_1\mathbf{e}^{2t} + c_2 t\mathbf{e}^{2t}$$

$$y'(t) = 2c_1\mathbf{e}^{2t} + c_2\mathbf{e}^{2t} + 2c_2 t\mathbf{e}^{2t}$$

Don't forget to product rule the second term! Plugging in the initial conditions gives the following system.

$$12 = y(0) = c_1$$

$$-3 = y'(0) = 2c_1 + c_2$$

This system is easily solve to get $c_1 = 12$ and $c_2 = -27$. The actual solution to the IVP is then.

$$y(t) = 12\mathbf{e}^{2t} - 27t\mathbf{e}^{2t}$$

REDUCTION OF ORDER

We are now going to look at the solution for non-constant coefficient second order DE of the form:

$$a(x)y''+b(x)y'+c(x)y=0$$

With a(x), *b(x)* and *c(x)* are now non-constants.

In general, finding solutions to these kinds of DE can be much more difficult than finding solutions to constant coefficient differential equations.
However, if we already know one solution to the differential equation we can use the method that is called reduction of order.

Let’s take a quick look at an example to see how this is done.

Example:
Find the general solution:

$$2t^2y''+ty'-3y = 0$$ given that $y_1(t) = t^{-1}$ is a solution.

Solution:

Reduction of order requires that a solution already be known. Without this known solution we won't be able to do reduction order. We will then assume the second solution is:

$$y_2(t) = v(t)y_1(t)$$

Thus, the second solution and the derivatives needed are:

$$y_2(t) = t^{-1}v \qquad y_2'(t) = -t^{-2}v + t^{-1}v' \qquad y_2''(t) = 2t^{-3}v - 2t^{-2}v'$$

Plugging these into the DE gives:

$$2t^2\left(2t^{-3}v - 2t^{-2}v' + t^{-1}v''\right) + t\left(-t^{-2}v + t^{-1}v'\right) - 3\left(t^{-1}v\right) = 0$$

$$2tv'' + (-4+1)v' + \left(4t^{-1} - t^{-1} - 3t^{-1}\right)v = 0$$

$$2tv'' - 3v' = 0$$

We will solve this by making the following **change of variable** to reduce the order.

Example: cont'd

$2tv''-3v'=0$ We change the variables to $w=v' \Rightarrow w'=v''$

Thus it becomes:

$2tw'-3w=0$

We've managed to reduce a second order DE down to a first order DE. It is left for you to get the solution of this DE as:

$$w(t)=c.t^{3/2}$$

Then recall our change of variable:

$$v'=w \quad \Rightarrow \quad v(t)=\int w\,dt=\int ct^{3/2}\,dt=\frac{2}{5}ct^{5/2}+k$$

For simplicity, We can write:

$$v(t)-c_1t^{5/2}+k \quad \Rightarrow \quad y_2(t)-(c_1t^{5/2}+k)t^{-1}-c_1t^{3/2}+kt^{-1}$$

Then the general solution is: $y=y_1(t)+y_2(t)=t^{-1}+c_1t^{3/2}+kt^{-1}$

$$y=c_1t^{3/2}+c_2t^{-1} \quad \#$$

REDUCTION OF ORDER

Short method for reduction of order

For the given ODE:

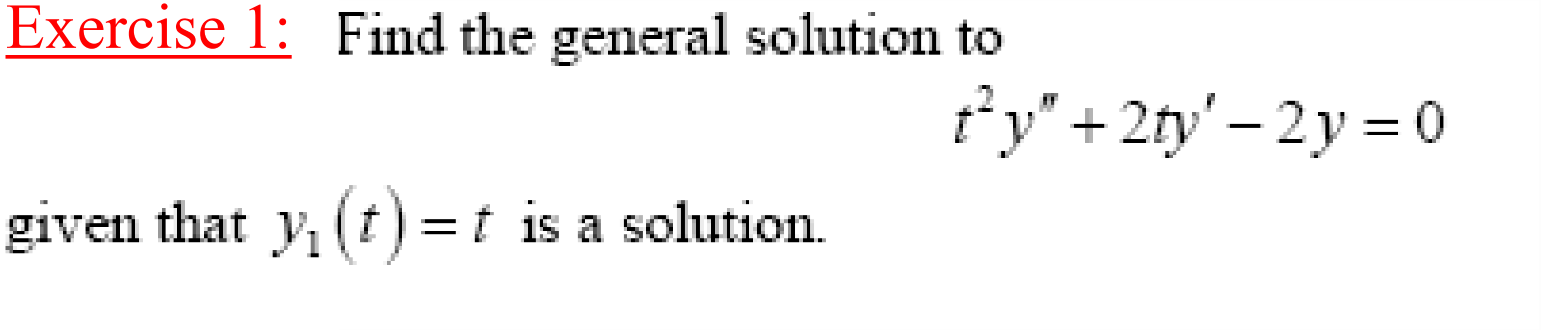
$$y''+p(x)y'+q(x)y=0$$

where y_1 is known. Then y_2 can be determined by: $y_2 = vy_1$

And v can be calculated by:

$$v' = \frac{e^{-\int p(x)dx}}{y_1^2} \longrightarrow \boxed{v = \int \frac{e^{-\int p(x)dx}}{y_1^2} dx}$$

Exercise 1:



Find the general solution to

$$t^2 y'' + 2ty' - 2y = 0$$

given that $y_1(t) = t$ is a solution.

Exercise 2:

It is known $y_1=x^2$ is a solution of $x^2y''-3xy'+4y=0$.

Find its general solution: $y=c_1y_1+c_2y_2$.

Thank You
To be continue…